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There exist infinitely many two component links which are 2-universal

Enrique Ramírez-Losada

Centro de Investigación en Matemáticas, AC, Guanajuato, Gto. México 36240, Mexico

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Abstract

We give a family of two component links which are 2-universal.

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1. Introduction

We are interested in the problem of finding 2-universal links. A link or knot l is *2-universal* if every closed orientable 3-manifold is a covering of S^3 branched along l, and all branching indices are one or two. If $f : M^3 \to S^3$ is a covering branched along l, the preimage of a meridian μ consists of components which map onto μ as unbranched coverings of degree $i_1, \dots, i_n$. These numbers are the *branching indices* of the component of l having meridian μ. A branched covering $f : M^3 \to S^3$ is of *type* $\{1,2\}$ if all branching indices are one or two.

The interest in 2-universal links or knots was pointed out by Thurston [6]; but the problem of finding 2-universal links or knots seems to be very difficult, for most known universal knots (links) are two bridge knots or Montesinos', and these are not 2-universal

E-mail address: kikis@cimat.mx (E. Ramírez-Losada).

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[1,3]. In fact, there is no example of a 2-universal knot despite the existence of 2-universal knots [1]. Several examples of 2-universal links are given in [2], however all these examples have at least three components. In this paper we give a family of two component links which are 2-universal.

The paper is arranged as follows: In Section 2 we give a covering $f: S^3 \to S^3$ branched along a torus knot $\tau_{p,q}$, $p = (q+1)/3$, $q = 2^{2n+1}$, $n \geqslant 1$, of type $\{1,2\}$. This covering will be very useful for the construction of the family of 2-universal links. In Section 3 we state and prove our main result.

2. Branched coverings of S^3

In this section we will show that S^3 is a covering of S^3 branched along the knot $\tau_{p,q}$ of type $\{1,2\}$. We will think of $\tau_{p,q}$ as a regular fiber of the Seifert bundle $(S^3, \tau_{p,q}) = (O, 0; (p-1)/p, -(q-3)/q)$.

Proposition 2.1. *Let $\tau_{p,q}$ be a torus knot in the 3-sphere with $p = (q+1)/3, q = 2^{2n+1}$, and $n \geqslant 1$. Then S^3 is a covering of type $\{1,2\}$ of S^3 branched along $\tau_{p,q}$.*

Proof. Let S be the base space of $(S^3, \tau_{p,q}) = (O, 0; (p-1)/p, -(q-3)/q)$ obtained by identifying each fiber to a point, and let $\rho: (S^3, \tau_{p,q}) \to S$ be the natural projection onto the base space. The restriction $\rho_0: (S^3 - \tau_{p,q}) \to (S - \rho(\tau_{p,q}))$ induces an epimorphism $\rho_{0*} of \pi_1(S^3 - \tau_{p,q})$ onto the orbifold fundamental group $\pi_1^o(S - \rho(\tau_{p,q})) = \mathbb{Z}_p * \mathbb{Z}_q = \|z_1, z_2\colon z_1^p = 1 = z_2^q\|$ with kernel the infinite cyclic group generated by a regular fiber h.

If $\|a_1, a_2\colon a_1^p = 1 = a_2^q\|$ is a presentation of $\pi_1(S^3 - \tau_{p,q})$, then we can write $\rho_{0*}(a_1) = z_1^q, \rho_{0*}(a_2) = z_2^p, \rho_{0*}(\mu) = z_1 z_2$, where μ is the meridian $\mu = a_1^{(p-1)} a_2^{-(q-3)}$. Now, consider the following representation: $\omega: \pi_1^o(S - \rho(\tau_{p,q})) \to S_{q+1}$ from the orbifold fundamental group into the symmetric group of $q+1$ elements, such that $\omega(z_1) = (1, 2, \ldots, p-1, p)(p+1, 2p+1, 2p+2, \ldots, q-1, q)(p+2, q+1, p+3, p+4, \ldots, 2p-1, 2p), \omega(z_2) = (1, q, q-1, \ldots, 3, 2)(q+1)$. Notice that $\omega(\rho_{0_*})$ is a transitive representation $\omega(\rho_{0_*}(h)) = id$, and $\omega(\rho_{0_*}(\mu)) = (p,q)(p+1, p+3)(p+2, q+1)$.

Let $f_1: M \to (S^3, \tau_{p,q})$ be the branched covering associated to $\omega(\rho_{0_*})$. By the Riemann–Hurwitz formula we compute the genus $\tilde{g}$ of the base space $\widetilde{S}$ of M and it is zero. Now, if s_p and s_q denote the exceptional fibers of order p and q in $(S^3, \tau_{p,q}) = (O, 0; (p-1)/p, -(q-3)/q)$, and x_p, x_q are the cone points of S corresponding to the exceptional fibers s_p and s_q, respectively, then the points $f_1^{-1}(x_p)$ and $f_1^{-1}(x_q)$ have ramification indices given by the cardinalities of the orbits of $\omega(z_i)$ $(i = 1, 2)$ [5]. So, M is the Seifert bundle:

$$\big(O, 0; (p-1)/1, (p-1/1), (p-1/1), -(q-3)/1, -(q-3)/q\big) = (O, 0; 3/q).$$

Note that this covering is of type $\{1,2\}$, and the preimage, $f_1^{-1}(\tau_{p,q})$, of the torus knot has $q-2$ components, three with branching index two, and branching index one for the others.

Now we take the 2-fold covering of $(O, 0; 3/q)$ branched along a component of $f_1^{-1}(\tau_{p,q})$ having branching index one. By [4, Lemma 2] we can see that this covering is the Seifert bundle $(O, 0; 3/(q/2))$. So, after composing, we obtain a branched covering $f_2 : (O, 0; 3/(q/2)) \to (S^3, \tau_{p,q})$. In a similar way, if we take the 2-fold covering of $(O, 0; 3/(q/2))$ branched along a preimage $f_2^{-1}(\tau_{p,q})$ with branching index one, we get [4, Lemma 2] the branched covering $f_3 : (O, 0; 3/(q/4)) \to (S^3, \tau_{p,q})$. Repeating this procedure $(2n + 1)$-times, we obtain a branched covering $f_{2n+2} : (O, 0; 3/1) \to (S^3, \tau_{p,q})$.

Finally, we take the universal covering of the lens space $(O, 0; 3/1)$ to get the branched covering $f : S^3 = (O, 0; 1/1) \to (S^3, \tau_{p,q})$. It is not difficult to see that f restricted to one component of $f^{-1}(\tau_{p,q})$ is a 3-fold covering, and each component of $f^{-1}(\tau_{p,q})$ has either branching index two or one. In fact, there are $r = 2^{2n+1}(3) + \sum_{i=1}^{2n} 2^i$ branching index two components, and $s = 2^{2n+1}(q + 1) - 2r$ branching index one components. $\square$

3. The 2-universal links

In this section we will give infinitely many two components links which are 2-universal.

In the following, we suppose that $(S^3, \tau_{p,q})$ and $S^1 \times D^2$ are oriented. We denote by $\eta(\tau_{p,q})$ a tubular neighborhood of $\tau_{p,q}$ (p and q as in Section 2.1) and by $l_1 = k_1 \cup k_2$ the link in $S^1 \times D^2$ of Fig. 1. Let $j : S^1 \times D^2 \to \eta(\tau_{p,q})$ be an orientation preserving homeomorphism which takes $j(S^1 \times \{*\})$ to a fiber $h \in \partial\eta(\tau_{p,q}) = \partial(S^3 - \mathrm{int}(\eta(\tau_{p,q})))$ of the Seifert fibration $(O, 0; (p - 1)/p, -(q - 3)/q)$ of S^3, where $* \in \partial D^2$. We denote the link $j(l_1)$ by l.

Theorem 3.1. *The two component link l is a universal link of type* $\{1, 2\}$.

Proof. Let $f : S^3 = (O, 0; 1/1) \to (S^3, \tau_{p,q})$ be the covering space of Proposition 2.1, and let T be a tubular neighborhood of $\tau_{p,q}$ containing l in its interior (see Fig. 1). We denote

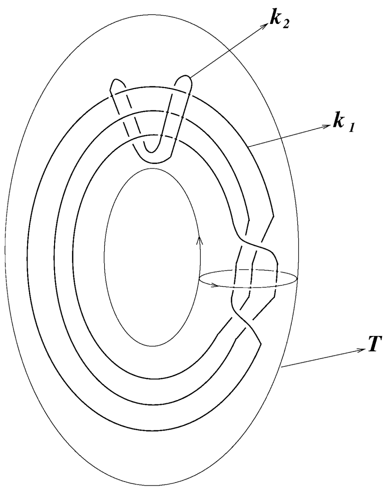

Fig. 1.

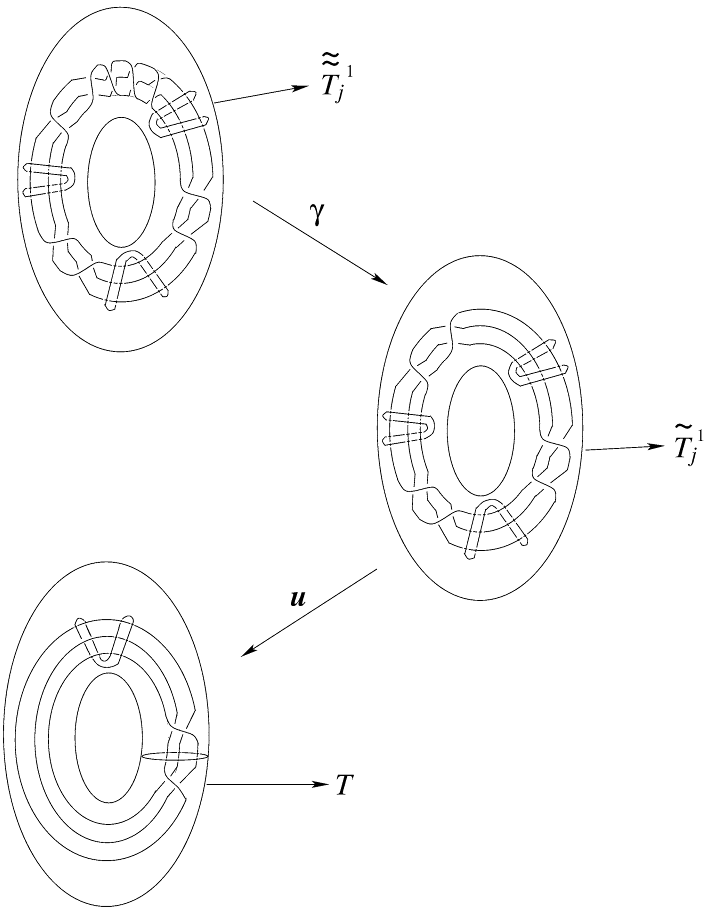

Fig. 2.

by $\widetilde{\widetilde{T}}{}^{2}$ each component of $f^{-1}(T)$ containing only one preimage of $\tau_{p,q}$ with branching index two ($i \in \{1, 2, \ldots, r\}$), and by $\widetilde{\widetilde{T}}{}_j^{1}$ each component containing one preimage of $\tau_{p,q}$ with branching index one, $j \in \{r+1, \ldots, s\}$. Notice that f restricted to $\widetilde{\widetilde{T}}{}_i^{2}$ is a covering of T branched along $\tau_{p,q}$. This covering factors as $\gamma : \widetilde{\widetilde{T}}{}_i^{2} \to \widetilde{T}_i \in u^{-1}(T)$, and $u : \widetilde{T}_i \to T$, where u is a 3-fold unbranched covering and γ is a 2-fold primitive covering of $\widetilde{T}_i$ branched along the core $\widetilde{C}$ of $\widetilde{T}_i$, i.e., $\gamma_* : \pi_1(\widetilde{\widetilde{T}}{}_i^{2}) \to \pi_1(\widetilde{T}_i)$ is surjective.

It is easy to see that $u^{-1}(l) \in \widetilde{T}_i$ is a six component link, three components coming from k_1 are isotopic to $\widetilde{C}$, for u is a 3-fold covering and k_1 is a monotone satellite of $\tau_{p,q}$ whose winding number is three (see Fig. 2).

For each component of $u^{-1}(T)$ containing a preimage of $\tau_{p,q}$ with branching index two, we define $g_i : \widetilde{T}_i \to \widetilde{T}_i$ to be the homeomorphism which maps one of the components of $u^{-1}(k_1)$ onto the core of $u^{-1}(T)$ and g_i restricted to $\partial \widetilde{T}_i$ is the identity. Define g to be

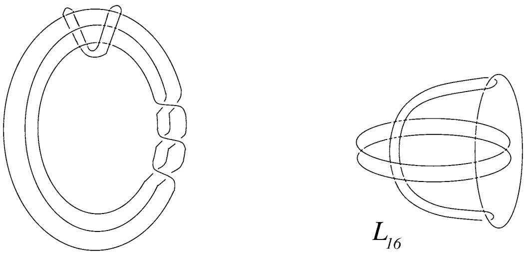

Fig. 3.

f on $S^3 - \bigsqcup \tilde{\tilde{\tilde{T}}}{}_i^2$ and $u \circ g_i^{-1} \circ \gamma$ on $\tilde{\tilde{\tilde{T}}}{}_i^2$. Hence $g : S^3 \to (S^3, l)$ is a covering branched along l of type $\{1, 2\}$, for the branching index is two for only one component of $g^{-1}(\tau_{p,q})$ in $\tilde{\tilde{\tilde{T}}}{}_i^2$ (recall that γ is a 2-fold branched covering) and one for the others.

Now, if we look at the preimage of $g^{-1}(l)$ contained in $\tilde{\tilde{\tilde{T}}}{}_j^2$ we see (Fig. 2) that this is a six component link (in the picture, the right full twist of $g^{-1}(l)$ is due to the Hopf fibration of $S^3 = (O, 0; 1/1)$). But this link contains a four component sublink which is isotopic to the link L_{16}, and this link is a 2-universal link [2]. Hence l is a 2-universal link, for each component of the sublink depicted in Fig. 3 has branching index one.

Corollary 3.1. *There exist infinitely many two component links which are* 2*-universal.*

References

[1] H.M. Hilden, M.T. Lozano, J.M. Montesinos, On 2-universal knots, Bol. Soc. Mat. Mexicana 10 (2004) 3.
[2] M.T. Lozano, J.M. Montesinos, Geodesics flows on hyperbolic orbifolds, and universal orbifolds, Pacific J. Math. 177 (1997) 109–147.
[3] V. Núñez, A note on 2-universal links, Bol. Soc. Mat. Mexicana 10 (2004) 3.
[4] V. Núñez, E. Ramírez-Losada, The trefoil knot is as universal as it can be, Topology Appl. 130 (2003) 1–17.
[5] E. Ramírez-Losada, S^3 as a cover of S^3 branched over a knot, Topology Appl. 121 (2002) 509–520.
[6] W. Thurston, Universal links, Preprint, 1982.